

THE GROWTH OF THE VORTICITY GRADIENT FOR THE TWO-DIMENSIONAL EULER FLOWS ON DOMAINS WITH CORNERS

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ABSTRACT. We consider the two-dimensional Euler equations in non-smooth domains with corners. It is shown that if the angle of the corner θ is strictly less than $\pi/2$, the Lipschitz estimate of the vorticity at the corner is at most single exponential growth and the upper bound is sharp. For the corner with the larger angle $\pi/2 < \theta < 2\pi$, $\theta \neq \pi$, we construct an example of the vorticity which loses continuity instantaneously. For the case $\theta \leq \pi/2$, the vorticity remains continuous inside the domain. We thus identify the threshold of the angle for the vorticity maintaining the continuity. For the borderline angle $\theta = \pi/2$, it is also shown that the growth rate of the Lipschitz constant of the vorticity can be double exponential, which is the same as in Kiselev-Sverak's result (Annals of Math., 2014).

1. INTRODUCTION

Let Ω be a two-dimensional domain. We are concerned with the Euler equations in Ω in the vorticity formulation:

$$(1.1) \quad \omega_t + (u \cdot \nabla)\omega = 0, \quad \omega(x, 0) = \omega_0(x).$$

Here ω is the fluid vorticity, and u is the velocity of the flow determined by the Biot-Savart law. We impose the no flow condition for the velocity at the boundary: $u \cdot n = 0$ on $\partial\Omega$, where n is the unit normal vector on the boundary. This implies the formula:

$$(1.2) \quad u(x, t) = \nabla^\perp \int_\Omega G_\Omega(x, y)\omega(y, t)dy,$$

where G_Ω is the Green function for the Dirichlet problem in Ω and $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$. The movement of a fluid particle, placed at a point $X \in \Omega$, is defined as the solution of the Cauchy problem

$$(1.3) \quad \frac{d\gamma_X(t)}{dt} = u(\gamma_X(t), t), \quad \gamma_X(0) = X,$$

and the vorticity ω is advected by

$$(1.4) \quad \omega(x, t) = \omega_0(\gamma_x^{-1}(t)).$$

Global regular solutions to the Euler equations (1.1) in smooth bounded domains were proved by Wolibner [11] and Hölder [3] and there are huge literature on this problem. Recently, there are growing interests in the study of (1.1) in nonsmooth domains. Existence of global weak solutions, with $u \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ and $\omega \in L^\infty(\mathbb{R}_+ \times \Omega)$, was proved by Taylor [10] for convex domains and by G  rard-Varet and Lacave [2] for more general (possibly not convex) domains. Uniqueness of the solution to the Euler equations (1.1) on domains with corners was shown by Lacave, Miot and Wang [8] for acute angles. For obtuse corners, Lacave [7] proved uniqueness

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of the solution under the assumption that the support of the vorticity never intersects the boundary. We are concerned with the question how fast the maximum of the gradient of the vorticity can grow as $t \rightarrow \infty$. When Ω is a smooth bounded domain, the best known upper bound on the growth is double-exponential [13], while the question whether such upper bound is sharp had been open for a long time. In 2014, Kiselev and Sverak [5] answered the question affirmatively for the case Ω is a disk. They gave an example of the solution growing with double exponential rate. For a general domain with C^3 -boundary see [12]. On the other hand, Kiselev and Zlatos [6] considered the 2D Euler flows on some bounded domain with certain cusps. They showed that the gradient of vorticity blows up at the cusps in finite time. These solutions are constructed by imposing certain symmetries on the initial data, which leads to a *hyperbolic flow scenario* near a stagnation point on the boundary. More precisely, by the hyperbolic flow scenario, particles on the boundary (near the stagnation point) head for the stagnation point for all time. Moreover the relation between this scenario and the geometry of the boundary plays a crucial role in the double exponential growth or the formation of the singularity. Thus it would be an interesting question to ask how the geometry of the boundary affects the growth of the solution. In [4] the authors considered the Euler equations (1.1) on the unit square and under a simple symmetry condition the growth of the Lipschitz constant of the vorticity on the boundary is shown to be at most single exponential at the stagnation point. In this paper, we are concerned with more general cases; the growth of the Lipschitz norm of the vorticity in bounded domains with general corners.

Definition 1.1. (i) Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain $0 < \theta < 2\pi$ with $\theta \neq \pi$. We say that $\partial\Omega$ has a corner of angle θ ($0 < \theta < 2\pi$) at $\xi \in \partial\Omega$, if there exist constants $r_0 > 0$ and $0 \leq \theta_0 < 2\pi$ such that, $\Omega \cap B(\xi, r_0) = \{x = (x_1, x_2) : \theta_0 < \arg(x - \xi) < \theta_0 + \theta\} \cap B(\xi, r_0)$. (ii) Let Ω be a domain with corners given in (i). We say Ω is symmetric with respect to the corner if $\theta_0 = -\frac{\theta}{2}$ and Ω is symmetric along the x_1 -axis.

Without loss of generality, by translation, rotation and scaling, we may assume that

$$(1.5) \quad \begin{cases} \text{diam}(\Omega) < 1 \text{ and } 0 \in \partial\Omega, \\ \partial\Omega \text{ has a corner of angle } \theta \text{ at } 0 \text{ with } \theta_0 = 0 \text{ in Definition 1.1.} \end{cases}$$

We now focus on the growth of the Lipschitz constant of ω with $a \in \overline{\Omega}$

$$\sup_{x \in \overline{\Omega}} \frac{|\omega(x, t) - \omega(a, t)|}{|x - a|}.$$

Our first result concerns the domain with the corner with the angle $\theta \leq \pi/2$.

Theorem 1.2. *Let Ω be a simple connected domain satisfying (1.5) and ω_0 be a Lipschitz function.*

(a) *For $0 < \theta < \frac{\pi}{2}$, there exists a constant $C > 0$ depending only on Ω such that*

$$(1.6) \quad \sup_{x \in \Omega} \frac{|\omega(x, t) - \omega(0, t)|}{|x|} \leq \|\omega_0\|_{\text{Lip}} e^{C\|\omega_0\|_{\infty} t} \quad \text{for } t > 0.$$

Moreover there exist an initial data ω_0 and a constant $C > 0$ such that

$$(1.7) \quad \sup_{x \in \Omega} \frac{|\omega(x, t) - \omega(0, t)|}{|x|} \geq C e^{Ct} \quad \text{for } t > 0.$$

(b) For $\theta = \frac{\pi}{2}$, there exists an initial data ω_0 with $\|\omega_0\|_{\text{Lip}} > 1$ such that

$$(1.8) \quad \sup_{x \in \Omega} \frac{|\omega(x, t) - \omega(0, t)|}{|x|} \geq \|\omega_0\|_{\text{Lip}}^{C \exp(Ct)} \quad \text{for } t > 0.$$

(c) If $\partial\Omega$ is $C^{1,1}$ except at $0 \in \partial\Omega$, then there exists a constant C depending only on Ω such that

$$(1.9) \quad |\omega(x, t) - \omega(y, t)| \leq \|\omega_0\|_{\text{Lip}} |x - y|^{\exp(-C\|\omega_0\|_{\infty}t)} \quad \text{for } x, y \in \Omega \text{ and } t > 0.$$

Remark 1.3. The assertion (a) shows that if $\theta < \frac{\pi}{2}$, then the growth of the Lipschitz constant at the corner of the vorticity is at most single exponential and the upper bound is sharp. For the case $\theta = \frac{\pi}{2}$, one can see from (b) that there exists an initial data ω_0 such that the growth of the Lipschitz constant of the vorticity at the corner is at least double-exponential. In our argument, we are imposing an infimum condition to the initial vorticity: $\inf_{x \in \Omega} \omega_0 > 0$ (see Lemma 3.3). This condition makes the proof simpler. Indeed, we do not need a bootstrapping argument as in the proof of [5, Theorem 1.1] anymore. The assertion (c) shows that the vorticity remains continuous in Ω although the Hölder exponent is decreasing in t . It is likely that the solution is Lipschitz continuous in Ω and the growth is at most exponential. We would like to address this issue elsewhere.

Remark 1.4. For the case $\theta = \frac{\pi}{2}$, we could not figure out whether the upper bound is indeed double-exponential. In fact we are analyzing local behavior of the flow near the corner by using the conformal mapping and the Green function of the unit upper half-disk. For smooth domains, it follows $C^{1,\alpha}$ -regularity of the velocity on $\overline{\Omega}$, we can obtain the double exponential upper bound without using conformal mappings. See [5, Theorem 2.1 and Proposition 2.2] for example.

Remark 1.5. Assume Ω is symmetric with respect to the corner and $\omega_0(x_1, x_2) = -\omega_0(x_1, -x_2)$ in $x \in \Omega$. Then, by Theorem 1.2, we can immediately see that if $\theta \in (0, \pi)$, then its corresponding solution has also single exponential bound. In this point of view, Theorem 1.2 can be considered as a generalization of [4]. To obtain the upper bound, we split the domain Ω into $\Omega \cap \{x : x_2 > 0\}$, and just apply Theorem 1.2 to the splitted domain (with the half angle $\theta/2$ case). In this case we do not need the infimum condition $\inf_{x \in \Omega} \omega_0(x) > 0$ (see Lemma 3.3) anymore.

We next consider the case $\theta > \pi/2$. In this case, we will see that the vorticity can lose continuity instantaneously.

Theorem 1.6. *Let Ω be a simply connected bounded domain satisfying (1.5). If $\pi/2 < \theta < \pi$, there are an initial data $\omega_0 \in C(\overline{\Omega})$ and its solution ω such that $\omega(t)$ instantaneously loses continuity in space. Furthermore, if $\pi < \theta < 2\pi$ and Ω is symmetric with respect to the corner, there also exist $\omega_0 \in C(\overline{\Omega})$ and its solution ω such that $\omega(t)$ instantaneously loses continuity.*

In the proof of our results, the estimates of the velocity fields near the corner play important roles as in [5, 12, 4]. One of the new ingredients in our proof is to use of the conformal mapping which have not used for the large time behavior of the vorticity. This enables us to obtain the explicit representation of the Green function G_Ω in the Biot-Savart law via the conformal mapping and to estimate the behavior of the velocity fields near the corner. Finally, we note that Theorems 1.2 and 1.6 hold for domains with more general corners or even finite number of corners; see Remark 2.4.

We use the following notation. By the symbol C we denote an absolute positive constant whose value is unimportant and may change from one occurrence to the next. If necessary, we

use C_0, C_1, \dots , to specify them. We say that f and g are comparable and write $f \approx g$ if two positive quantities f and g satisfies $C^{-1} \leq f/g \leq C$ with some constant $C \geq 1$. The constant C is referred to as the constant of comparison. We have to pay attention for the dependency of the constant of comparison. For $x = (x_1, x_2) \in \mathbb{R}^2$ we let $\tilde{x} = (-x_1, x_2)$, $\bar{x} = (x_1, -x_2)$ and $x^* = x/|x|^2$. Let m be the two-dimensional Lebesgue outer measure.

2. PRELIMINARIES

Let D be a bounded simply connected open subset of \mathbb{R}^2 . Identifying \mathbb{R}^2 with \mathbb{C} , the Riemann mapping theorem states that there exists a conformal mapping f of the open unit disk $D = B(0, 1)$ onto D . Moreover Carathéodory theorem asserts that if D is Jordan domain, then f has a continuous injective extension to \bar{D} . If D is $C^{1,1}$ -domain, then the following Kellogg-Warschawski theorem holds. See [9, Theorem 3.6] and [1, Theorem II.4.3 and Lemma II.4.4].

Theorem 2.1. *Let f be a conformal map from D onto a $C^{1,1}$ -domain D . Then f' has a continuous extension to \bar{D} and*

$$(2.1) \quad \frac{f(\zeta) - f(z)}{\zeta - z} \rightarrow f'(z) \neq 0 \quad \text{for } \zeta \rightarrow z, \quad \zeta, z \in \bar{D},$$

$$(2.2) \quad |f'(z_1) - f'(z_2)| \leq C|z_1 - z_2| \log |z_1 - z_2|^{-1} \quad \text{for } z_1, z_2 \in \bar{D}, \quad |z_1 - z_2| < 1.$$

The following theorem states the smoothness of conformal map $f : D \rightarrow D$ in a neighborhood of $f^{-1}(\zeta)$ depends only on the smoothness of $\partial\Omega$ in a neighborhood of $\zeta \in \partial\Omega$. See [1, Theorem II.4.1].

Theorem 2.2. *Let D_1 and D_2 be Jordan domains such that $D_1 \subset D_2$ and let $\gamma \subset \partial\Omega_1 \cap \partial\Omega_2$ be an open subarc. Let φ_j be a conformal map of D onto D_j ($j = 1, 2$). Then $\psi = \varphi_2^{-1} \circ \varphi_1$ has an analytic continuation across $\varphi_1^{-1}(\gamma)$, and $\psi' \neq 0$ on $\varphi_1^{-1}(\gamma)$.*

Let $U = \{z = x + iy \in D : y > 0\}$. Using the above theorems, we show the following lemma.

Lemma 2.3. *Assume that Ω satisfies (1.5). Let $\beta = \pi/\theta$. Then there exists a conformal map $f : \Omega \rightarrow U$ with $f(0) = 0$. Let $g \equiv f^{-1}$. Moreover there exist a constant $\delta_0 > 0$ such that*

- (i) $|f(z)| \approx |z|^\beta$ and $|f'(z)| \approx |z|^{\beta-1}$ for $z \in \bar{\Omega} \cap B(0, \delta_0)$,
- (ii) $|g(w)| \approx |w|^{\frac{1}{\beta}}$ and $|g'(w)| \approx |w|^{\frac{1}{\beta}-1}$ for $w \in \bar{U} \cap B(0, \delta_0)$.

Here δ_0 and the constant C_0 of comparison depend only on Ω .

Proof. Let $\varphi(z) = z^\beta$. Observe that $\varphi(0) = 0$ and $\partial\varphi(\Omega)$ is locally a straight line near 0. By the Riemann mapping theorem, there exists a conformal map $f_1 : \varphi(\Omega) \rightarrow U$ with $f_1(0) = 0$. Let $g_1 = f_1^{-1}$. The Kellogg-Warschawski theorem and Theorem 2.2 imply that there is a constant $\delta > 0$ such that

$$|f_1'(z)| \approx 1, \quad |g_1'(w)| \approx 1, \quad |g_1(w)| \approx |w|$$

for $z \in \overline{\varphi(\Omega)} \cap B(0, \delta)$ and $w \in \bar{U} \cap B(0, \delta)$. Let $f = f_1 \circ \varphi$ and let $\delta_0 < \delta$ be a sufficiently small constant. Then we see that $f(0) = 0$. Since $f'(z) = f_1'(\varphi(z))\varphi'(z)$ and $g'(w) = \varphi^{-1}(g_1(w))g_1'(w)$, we have

$$(2.3) \quad |f'(z)| \approx |\varphi'(z)| \approx |z|^{\beta-1}$$

and

$$(2.4) \quad |g'(w)| \approx |(\varphi^{-1})(g_1(w))| \approx |g_1(w)|^{\frac{1}{\beta}-1} \approx |w|^{\frac{1}{\beta}-1}$$

for $z \in \overline{\varphi(\Omega)} \cap B(0, \delta_0)$ and $w \in \overline{U} \cap B(0, \delta_0)$. It follows from the above estimates and the mean-value property with $f(0) = 0$ and $g(0) = 0$ that

$$|f(z)| \approx |z|^\beta, \quad |g(w)| \approx |w|^{\frac{1}{\beta}}$$

for $z \in \overline{\varphi(\Omega)} \cap B(0, \delta_0)$ and $w \in \overline{U} \cap B(0, \delta_0)$. Thus the properties (i),(ii) hold. \square

Remark 2.4. Alternatively, we claim that Theorems 1.2 and 1.6 hold for domains with a more general corners. Let $\gamma(s)$ be a parametrization of $\partial\Omega$ with $\gamma(0) = 0$. We consider a domain such that γ is $C^{1,1}$ -Jordan curve except at $0 \in \partial\Omega$ and $\lim_{s \searrow 0} \arg \gamma(s) - \gamma(-s) = \theta$. In the proof of Lemma 2.3, we would also need a condition

$$(2.5) \quad (\gamma(s))^\beta \text{ is } C^{1,1} \text{ close to } 0 \in \partial\Omega,$$

in order to use the Kellogg-Warschawski theorem. Then Lemma 2.3 holds for domains with a general corner. For simplicity, we assume that Ω satisfies (1.5).

3. THE KEY LEMMAS

To prove Theorems 1.2 and 1.6, we need a technical lemma for the expansion of velocity field. Assume that Ω satisfies (1.5). Since the Green function for the unit upper half-disk U is given explicitly by

$$G_U(x, y) = \frac{1}{2\pi} (\log |x - y| - \log |x - y^*| - \log |\bar{x} - y| + \log |\bar{x} - y^*|),$$

the Green function for Ω is given explicitly by

$$\begin{aligned} G_\Omega(x, y) &= G_U(f(x), f(y)) \\ &= \frac{1}{2\pi} (\log |f(x) - f(y)| - \log |f(x) - f(y)^*| - \log |\overline{f(x)} - f(y)| + \log |\overline{f(x)} - f(y)^*|), \end{aligned}$$

where f is the conformal map of Ω onto U in Lemma 2.3. Let

$$G(x, y) = \log \frac{|f(x) - f(y)|}{|\overline{f(x)} - f(y)|}, \quad G^*(x, y) = \log \frac{|\overline{f(x)} - f(y)^*|}{|f(x) - f(y)^*|}$$

for $x, y \in \Omega$. Firstly we get an upper bound of u near the corner.

Lemma 3.1. *Let $0 < \theta < \pi$ and $\beta = \pi/\theta$. Assume that Ω satisfies (1.5). There exists a constant $C > 0$ depending only on Ω such that*

$$(3.1) \quad |u(x, t)| \leq C \|\omega_0\|_\infty \begin{cases} |x| & \text{if } \beta > 2, \\ |x| \log |x|^{-1} & \text{if } \beta = 2, \\ |x|^{\beta-1} & \text{if } 1 < \beta < 2, \end{cases}$$

for $x \in \Omega$ and $t > 0$. In particular, we see that $u(0, t) = 0$ for any $t > 0$.

Proof. Let f, g, C_0 and δ_0 be as in Lemma 2.3. Let δ be a small positive constant to be determined later and let $x \in \Omega$. It is sufficient to show that (3.1) for $|x| < \delta$. By Lemma 2.3(i), we observe that there exist $0 < \varepsilon < \delta_0$ such that if $y \in \Omega_+$ and $|y| \geq \varepsilon$, then $|f(y)| \geq \left(\frac{\varepsilon}{C_0}\right)^\beta$. Let us to be $\delta < \frac{\varepsilon}{(2C_0^2)^\beta}$. Then we have

$$(3.2) \quad u_1(x, t) = \frac{1}{2\pi} \int_{\Omega \cap B(0, \varepsilon)} \frac{\partial G}{\partial x_2}(x, y) \omega(y, t) dy + \frac{1}{2\pi} \int_{\Omega \setminus B(0, \varepsilon)} \frac{\partial G}{\partial x_2}(x, y) \omega(y, t) dy + \frac{1}{2\pi} \int_{\Omega} \frac{\partial G^*}{\partial x_2}(x, y) \omega(y, t) dy.$$

for $|x| < \delta$.

Firstly we estimate the last term of the right hand side of (3.2). Assume that $\delta < \frac{1}{(2C_0)^\beta}$. Since $|f(y)^*| \geq 1$, we have $|f(x)| \leq C_0|x|^\beta \leq \frac{1}{2} \leq \frac{1}{2}|f(y)^*|$, so that

$$(3.3) \quad |\overline{f(x)} - f(y)^*| \geq |f(x) - f(y)^*| \geq \frac{1}{2}|f(y)^*| \geq \frac{1}{2}.$$

We have

$$\frac{\partial G^*}{\partial x_2}(x, y) = -\frac{(f_1(x) - f_1^*(y))\frac{\partial f_1}{\partial x_2}(x) + (f_2(x) - f_2^*(y))\frac{\partial f_2}{\partial x_2}(x)}{|f(x) - f(y)^*|^2} + \frac{(f_1(x) - f_1^*(y))\frac{\partial f_1}{\partial x_2}(x) + (f_2(x) + f_2^*(y))\frac{\partial f_2}{\partial x_2}(x)}{|\overline{f(x)} - f(y)^*|^2},$$

where $f(x) = (f_1(x), f_2(x))$ and $f(y)^* = (f_1^*(y), f_2^*(y))$. Thus for $y \in \Omega_+$, we have

$$\left| \frac{\partial G^*}{\partial x_2}(x, y) \right| \leq C \frac{|f'(x)|}{|f(x) - f(y)^*|} \leq C|x|^{\beta-1},$$

by (3.3) and Lemma 2.3(i). Therefore we have

$$(3.4) \quad \left| \int_{\Omega} \frac{\partial G^*}{\partial x_2}(x, y) \omega(y, t) dy \right| \leq C|x|^{\beta-1} \|\omega_0\|_{\infty}.$$

Next we estimate the second term of the right hand side of (3.2). Let $y \in \Omega \setminus B(0, \varepsilon)$. Assume that $\delta < \frac{\varepsilon}{(2C_0)^{\frac{1}{\beta}} C_0}$. Then $|f(x)| \leq C_0|x|^\beta \leq \frac{1}{2} \left(\frac{\varepsilon}{C_0}\right)^\beta \leq \frac{1}{2}|f(y)|$, so that

$$|\overline{f(x)} - f(y)| \geq |f(x) - f(y)| \geq \frac{1}{2}|f(y)| \geq \frac{1}{4} \left(\frac{\varepsilon}{C_0}\right)^\beta.$$

Since

$$\frac{\partial G}{\partial x_2}(x, y) = +\frac{(f_1(x) - f_1(y))\frac{\partial f_1}{\partial x_2}(x) + (f_2(x) - f_2(y))\frac{\partial f_2}{\partial x_2}(x)}{|f(x) - f(y)|^2} - \frac{(f_1(x) - f_1(y))\frac{\partial f_1}{\partial x_2}(x) + (f_2(x) + f_2(y))\frac{\partial f_2}{\partial x_2}(x)}{|\overline{f(x)} - f(y)|^2},$$

we have

$$\left| \frac{\partial G}{\partial x_2}(x, y) \right| \leq C \frac{|f'(x)|}{|f(x) - f(y)|} \leq C|x|^{\beta-1},$$

by Lemma 2.3(i). Therefore we have

$$(3.5) \quad \left| \int_{\Omega \setminus B(0, \varepsilon)} \frac{\partial G}{\partial x_2}(x, y) \omega(y, t) dy \right| \leq C|x|^{\beta-1} \|\omega_0\|_{\infty}.$$

Finally we consider the first term of the right hand side of (3.2). In this case, the singularity at $x = y$ appears. So we need to calculate more carefully. We have

$$\left| \int_{\Omega \cap B(0, \varepsilon)} \frac{\partial G}{\partial x_2}(x, y) \omega(y, t) dy \right| \leq C|x|^{\beta-1} \|\omega_0\|_{\infty} \int_{\Omega \cap B(0, \varepsilon)} \frac{dy}{|f(x) - f(y)|}$$

by Lemma 2.3(i). Let $z = f(x)$ and $\varepsilon' = C_0 \varepsilon^\beta$. The substitution $w = f(y)$ yields

$$\begin{aligned}
\int_{\Omega \cap B(0, \varepsilon)} \frac{dy}{|f(x) - f(y)|} &\leq \int_{U \cap B(0, \varepsilon')} \frac{dw}{|z - w| |f'(y)|^2} \\
&\leq C \int_{B(0, \varepsilon')} \frac{dw}{|z - w| |w|^{2-2/\beta}} \\
&\leq C \left(\int_{B(0, \frac{1}{2}|f(x)|)} \frac{dw}{|z - w| |w|^{2-2/\beta}} + \int_{B(z, \frac{1}{2}|f(x)|)} \frac{dw}{|z - w| |w|^{2-2/\beta}} \right. \\
&\quad \left. + \int_{B(0, \varepsilon') \setminus [B(0, \frac{1}{2}|f(x)|) \cup B(z, \frac{1}{2}|f(x)|)]} \frac{dw}{|z - w| |w|^{2-2/\beta}} \right) \\
&\leq C \left(|x|^{-\beta} \int_0^{\frac{1}{2}|f(x)|} r^{\frac{2}{\beta}-1} dr + |x|^{2-2\beta} \int_0^{\frac{1}{2}|f(x)|} dr + \int_{\frac{1}{2}|f(x)|}^{\varepsilon'} r^{\frac{2}{\beta}-2} dr \right) \\
&\leq \begin{cases} C|x|^{2-\beta} & \text{if } \beta > 2, \\ C \log |x|^{-1} & \text{if } \beta = 2, \\ C & \text{if } 1 < \beta < 2, \end{cases}
\end{aligned}$$

by Lemma 2.3(i). Therefore we have

$$(3.6) \quad \left| \int_{\Omega \cap B(0, \varepsilon)} \frac{\partial G}{\partial x_2}(x, y) \omega(y, t) dy \right| \leq C \|\omega_0\|_\infty \begin{cases} |x| & \text{if } \beta > 2, \\ |x| \log |x|^{-1} & \text{if } \beta = 2, \\ |x|^{\beta-1} & \text{if } 1 < \beta < 2. \end{cases}$$

Combining (3.4), (3.5) and (3.6), we have

$$|u_1(x, t)| \leq C \|\omega_0\|_\infty \begin{cases} |x| & \text{if } \beta > 2, \\ |x| \log |x|^{-1} & \text{if } \beta = 2, \\ |x|^{\beta-1} & \text{if } 1 < \beta < 2. \end{cases}$$

In a similar way to the proof of this estimate, we obtain that

$$|u_2(x, t)| \leq C \|\omega_0\|_\infty \begin{cases} |x| & \text{if } \beta > 2, \\ |x| \log |x|^{-1} & \text{if } \beta = 2, \\ |x|^{\beta-1} & \text{if } 1 < \beta < 2, \end{cases}$$

so that (3.1) holds. \square

If $\theta \leq \frac{\pi}{2}$ and $\partial\Omega$ is $C^{1,1}$ except at $0 \in \partial\Omega$, then the velocity u is log-Lipchitz continuous on $\overline{\Omega}$. In a way similar to the proof of [8, Proposition 3.4] we obtain the following lemma.

Lemma 3.2. *Let $0 < \theta \leq \frac{\pi}{2}$. Assume that Ω satisfies (1.5) and Ω is $C^{1,1}$ except at $0 \in \partial\Omega$. There exists a constant $C > 0$ depending only on Ω such that*

$$(3.7) \quad |u(x, t) - u(y, t)| \leq C \|\omega_0\|_\infty |x - y| \log |x - y|^{-1}$$

for $x, y \in \Omega$ and $t > 0$.

Next we get a lower bound of u near the corner.

Lemma 3.3. *Assume that Ω satisfies (1.5). Let $\beta = \pi/\theta$.*

(a) *Let $0 < \theta \leq \frac{\pi}{2}$. If $c_0 = \min_{x \in \Omega} \omega_0 > 0$, then there exist constants $\delta_1 > 0$ and $C_1 > 0$*

depending only on Ω , $\|\omega_0\|_\infty$ and c_0 such that

$$(3.8) \quad u_1(x, t) \leq -C_1 \begin{cases} x_1 & \text{if } 0 < \theta < \frac{\pi}{2}, \\ x_1 \log x_1^{-1} & \text{if } \theta = \frac{\pi}{2}, \end{cases}$$

for $x = (x_1, 0) \in \partial\Omega$, $0 < x_1 < \delta_1$ and $t > 0$.

(b) Let $\frac{\pi}{2} < \theta < \pi$. If $\omega_0 > 0$ then there exist constants $\delta_2 > 0$ and $C_2 > 0$ depending only on Ω and $\|\omega_0\|_\infty$ such that

$$(3.9) \quad u_1(x, t) \leq -C_2 x_1^{\beta-1}$$

for $x = (x_1, 0) \in \partial\Omega$, $0 < x_1 < \delta_2$ and $t > 0$.

Proof. Let f be as in Lemma 2.3. Let δ be a small positive constant. Now we consider the particle behavior on the boundary. Let $x = (x_1, 0) \in \partial\Omega$, $0 < x_1 < \delta$. Observe that $f_2(x) = 0$, $\partial f_2 / \partial x_1(x) = 0$ and $\partial f_2 / \partial x_2(x) \geq C|x|^\beta \geq Cx_1^{\beta-1}$.

(a) Assume that $0 < \theta \leq \frac{\pi}{2}$ and $c_0 = \min_{y \in \Omega} \omega_0 > 0$. Since (1.4), we see that $\min_{y \in \Omega} \omega(y, t) = c_0$ for any $t > 0$. Let ε be as in the proof of Lemma 3.1. By (3.4) and (3.5), we have

$$\begin{aligned} u_1(x, t) &\leq -\frac{1}{\pi} \frac{\partial f_2}{\partial x_2}(x) \int_{\Omega \cap B(0, \varepsilon)} \frac{f_2(y)}{|f(x) - f(y)|^2} \omega(y, t) dy + C\|\omega_0\|_\infty |x|^{\beta-1} \\ &\leq -Cx_1^{\beta-1} \int_{\Omega \cap B(0, \varepsilon)} \frac{f_2(y)}{|f(x) - f(y)|^2} dy + C\|\omega_0\|_\infty Cx_1^{\beta-1}. \end{aligned}$$

Let $z = f(x)$ and $\varepsilon' = C_0^{-1} \varepsilon^\beta$. The substitution $w = f(y)$ yields

$$\begin{aligned} \int_{\Omega \cap B(0, \varepsilon)} \frac{f_2(y)}{|f(x) - f(y)|^2} dy &\geq C \int_{U \cap B(0, \varepsilon')} \frac{w_2}{|z - w|^2 |w|^{2-2/\beta}} dw \\ &\geq C \int_{2|z|}^{\varepsilon'} \int_0^\pi \frac{\sin \theta}{r^{2-2/\beta}} d\theta dr \\ &\geq C \int_{2|z|}^{\varepsilon'} r^{-2+2/\beta} dr \\ &\geq C \begin{cases} x_1^{-\beta+2} & \text{if } \beta > 2, \\ \log x_1^{-1} & \text{if } \beta = 2, \end{cases} \end{aligned}$$

by Lemma 2.3(i). Therefore we obtain

$$u_1(x, t) \leq -C \begin{cases} x_1(1 - Cx_1^{\beta-2}) & \text{if } \beta > 2, \\ x_1(\log x_1^{-1} - C) & \text{if } \beta = 2. \end{cases}$$

We can choose $\delta > 0$ sufficiently small so that (3.8) holds.

(b) Assume $\frac{\pi}{2} < \theta < \pi$ and $\omega_0 > 0$ on Ω . Since (1.4), we see that $\omega(y, t) > 0$ for any $y \in \Omega$ and $t > 0$. Note that $\overline{f(x)} = f(x)$, $f_2(x) = 0$ for $x_2 = 0$ and $f_1^*(y)f_2(y) = f_1(y)f_2^*(y)$, $f_2(y) =$

$f_2^*(y)|f(y)|^2, f_2^*(y) = f_2(y)|f(y)^*|^2$. We obtain that

$$\begin{aligned} u_1(x) &= \frac{1}{\pi} \frac{\partial f_2}{\partial x_2}(x) \int_{\Omega} \left(-\frac{f_2(y)}{|f(x) - f(y)|^2} + \frac{f_2^*(y)}{|f(x) - f(y)^*|^2} \right) \omega(y, t) dy \\ &= \frac{1}{\pi} \frac{\partial f_2}{\partial x_2}(x) \int_{\Omega} \frac{-f_1(x)^2(f_2^*(y) - f_2(y)) - f_2(y)|f(y)^*|^2 + f_2^*(y)|f(y)|^2}{|f(x) - f(y)|^2|f(x) - f(y)^*|^2} \omega(y, t) dy \\ &= \frac{1}{\pi} \frac{\partial f_2}{\partial x_2}(x) \int_{\Omega} \frac{(f_2^*(y) - f_2(y))(f_1(x)^2 - 1)}{|f(x) - f(y)|^2|f(x) - f(y)^*|^2} \omega(y, t) dy \\ &\leq -C\chi_1^{\beta-1} \int_{\Omega} \frac{f_2^*(y) - f_2(y)}{|f(x) - f(y)^*|^2} \omega(y, t) dy, \end{aligned}$$

where $f_1(x)^2 - 1 \leq -C, f_2^*(y) - f_2(y) > 0, |f(x) - f(y)| \leq C$ and $\omega(y, t) > 0$ are used in the last inequality. For sufficiently small $r > 0$, we let

$$\Omega(r) = \{y \in \Omega : \text{dist}(y, \partial\Omega) < r\}.$$

If $y \in \Omega \setminus \Omega(r)$, then there exists a constant C such that

$$f_2^*(y) - f_2(y) \geq \frac{1}{C} \quad \text{and} \quad |f(x) - f(y)^*| \leq C,$$

since $f(y)$ is away from the origin. Then we have

$$\int_{\Omega} \frac{f_2^*(y) - f_2(y)}{|f(x) - f(y)^*|^2} \omega(y, t) dy \geq \int_{\Omega \setminus \Omega(r)} \frac{f_2^*(y) - f_2(y)}{|f(x) - f(y)^*|^2} \omega(y, t) dy \geq \int_{\Omega \setminus \Omega(r)} \omega(y, t) dy \geq C,$$

so that

$$u_1(x) \leq -C\chi_1^{\beta-1}.$$

Thus (3.9) holds. \square

4. PROOF OF THEOREMS 1.2 AND 1.6

In this section we will show Theorems 1.2 and 1.6.

Proof of Theorem 1.2. Let us consider the trajectory $\gamma_X(t) = (\gamma_X^1(t), \gamma_X^2(t))$ starting from a point $X \in \overline{\Omega}$. Let $x = \gamma_X(t)$.

(a) Let $0 < \theta < \frac{\pi}{2}$. Assume that ω_0 is Lipschitz. By Lemma 3.1 and (1.3), we have

$$\left| \frac{d}{dt} |\gamma_X(t)| \right| \leq \left| \frac{d\gamma_X(t)}{dt} \right| \leq C \|\omega_0\|_{\infty} |\gamma_X(t)| \quad \text{for all } t > 0,$$

and so

$$\frac{d}{dt} |\gamma_X(t)| \geq -C \|\omega_0\|_{\infty} |\gamma_X(t)| \quad \text{for all } t > 0.$$

By Gronwall's lemma we have $|\gamma_X(t)| \geq |X|e^{-C\|\omega_0\|_{\infty}t}$, so that $|\gamma_X^{-1}(t)| \leq |x|e^{C\|\omega_0\|_{\infty}t}$. Then we see that $\gamma_0^{-1}(t) = 0$. Since $\omega(x, t) = \omega_0(\gamma_x^{-1}(t))$ by the 2D Euler flows in the Lagrangian form, and ω_0

is Lipschitz, we obtain

$$\begin{aligned}
|\omega(x, t) - \omega(0, t)| &= |\omega_0(\gamma_x^{-1}(t)) - \omega_0(\gamma_0^{-1}(t))| \\
&= |\omega_0(\gamma_x^{-1}(t)) - \omega_0(0)| \\
&\leq \|\omega_0\|_{\text{Lip}} |\gamma_x^{-1}(t)| \\
&\leq \|\omega_0\|_{\text{Lip}} |x| e^{C\|\omega_0\|_{\infty} t}.
\end{aligned}$$

Thus (1.6) holds.

Next we consider an initial data ω_0 defined by

$$\omega_0(x) = |x| + 1.$$

Let δ_1 be as in Lemma 3.3(a). Due to the boundary condition on u , the trajectories which start at the boundary stay on the boundary for all times. We consider the trajectory starting from a point $X = (X_1, 0) \in \partial\Omega$ with $0 < X_1 < \delta_1$. Note that $\gamma_X^2(t) \equiv 0$ for any $t > 0$. By Lemma 3.3 and (1.3), we have

$$\frac{d\gamma_X^1(t)}{dt} \leq -C_1 \gamma_X^1(t) \quad \text{for all } t > 0.$$

By Gronwall's lemma we have $\gamma_X^1(t) \leq X_1 e^{C_1 t}$. We obtain that

$$\begin{aligned}
\sup_{x \in \overline{\Omega}, x \neq 0} \frac{|\omega(\gamma_X(t), t) - \omega(0, t)|}{|x|} &\geq \frac{|\omega(\gamma_X(t), t) - \omega(0, t)|}{|\gamma_X(t)|} \\
&= \frac{|\omega_0(X) - \omega_0(0)|}{\gamma_X^1(t)} \\
&\geq \frac{|\omega_0(X) - \omega_0(0)|}{X_1} e^{C_1 t} = e^{C_1 t}.
\end{aligned}$$

Thus (1.7) holds.

(b) For any $\varepsilon > 0$ we consider an initial data ω_0 defined by

$$\omega_0(x) = \min \left\{ \frac{|x|}{\varepsilon} + 1, 2 \right\}.$$

We see that $\|\omega_0\|_{\text{Lip}} = \varepsilon^{-1}$. Let δ_1 be as in Lemma 3.3. Note that δ_1 is independent of ε , instead depending on $\max \omega_0$ and $\min \omega_0$. Assume that $\varepsilon < \delta_1$. Due to the boundary condition on u , the trajectories which start at the boundary stay on the boundary for all times. We consider the trajectory starting from a point $X = (\varepsilon, 0) \in \partial\Omega$. Note that $\gamma_X^2(t) \equiv 0$ for any $t > 0$. By Lemma 3.3 and (1.3), we have

$$\frac{d\gamma_X^1(t)}{dt} \leq C_1 \gamma_X^1(t) \log \gamma_X^1(t) \quad \text{for all } t > 0.$$

By Gronwall's lemma we have $\gamma_X^1(t) \leq \varepsilon^{\exp(C_1 t)}$. We obtain that

$$\begin{aligned}
\sup_{x \in \overline{\Omega}, x \neq 0} \frac{|\omega(x, t) - \omega(0, t)|}{|x|} &\geq \frac{|\omega(\gamma_X(t), t) - \omega(0, t)|}{|\gamma_X(t)|} \\
&= \frac{|\omega_0(X) - \omega_0(0)|}{\gamma_X^1(t)} \\
&\geq \varepsilon^{-\exp(C_1 t)} = \|\omega_0\|_{\text{Lip}}^{\exp(C_1 t)}.
\end{aligned}$$

Thus (1.8) holds.

(c) Let $0 < \theta \leq \frac{\pi}{2}$. Assume that Ω is $C^{1,1}$ except at $0 \in \partial\Omega$ and ω_0 is Lipschitz. Let $\gamma_Y(t)$ starting from a point $Y \in \overline{\Omega}$. Let $y = \gamma_Y(t)$. By Lemma 3.2 and (1.3), we have

$$\left| \frac{d}{dt} |\gamma_X(t) - \gamma_Y(t)| \right| \leq \left| \frac{d}{dt} (\gamma_X(t) - \gamma_Y(t)) \right| \leq C \|\omega_0\|_\infty |\gamma_X(t) - \gamma_Y(t)| \log |\gamma_X(t) - \gamma_Y(t)|^{-1}$$

for all $t > 0$, and so

$$\frac{d}{dt} |\gamma_X(t) - \gamma_Y(t)| \geq C \|\omega_0\|_\infty |\gamma_X(t) - \gamma_Y(t)| \log |\gamma_X(t) - \gamma_Y(t)| \quad \text{for all } t > 0.$$

By Gronwall's lemma we have

$$|\gamma_X(t) - \gamma_Y(t)| \geq |X - Y|^{\exp(C\|\omega_0\|_\infty t)}.$$

Thus we obtain that

$$|\gamma_x^{-1}(t) - \gamma_y^{-1}(t)| \leq |x - y|^{\exp(-C\|\omega_0\|_\infty t)}.$$

Then we see that $\gamma_0^{-1}(t) = 0$. Since $\omega(x, t) = \omega_0(\gamma_x^{-1}(t))$ by the 2D Euler flows in the Lagrangian form, and ω_0 is Lipschitz, we obtain

$$\begin{aligned} |\omega(x, t) - \omega(y, t)| &= |\omega_0(\gamma_x^{-1}(t)) - \omega_0(\gamma_y^{-1}(t))| \\ &= |\omega_0(\gamma_x^{-1}(t)) - \omega_0(\gamma_y^{-1}(t))| \\ &\leq \|\omega_0\|_{\text{Lip}} |\gamma_x^{-1}(t) - \gamma_y^{-1}(t)| \\ &\leq \|\omega_0\|_{\text{Lip}} |x - y|^{\exp(-C\|\omega_0\|_\infty t)}. \end{aligned}$$

Thus (3.7) holds. \square

Proof of Theorem 1.6. Firstly we assume that $\frac{\pi}{2} < \theta < \pi$. Let us consider the trajectory $\gamma_X(t) = (\gamma_X^1(t), \gamma_X^2(t))$ starting from a point $X \in \Omega$. Let $x = \gamma_X(t)$. Now we consider a continuous initial data ω_0 defined by

$$\omega_0(x) = |x|.$$

Let δ_2 be as in Lemma 3.3. Due to the boundary condition on u , the trajectories which start at the boundary stay on the boundary for all times. We consider the trajectory starting from a point $X = (X_1, 0) \in \partial\Omega$ with $0 < X_1 < \delta_2$. By Lemma 3.3 and (1.3), we have

$$\frac{d\gamma_X^1(t)}{dt} \leq -C_2 \left(\gamma_X^1(t) \right)^{\beta-1} \quad \text{for all } t > 0.$$

By Gronwall's lemma we have

$$\gamma_X^1(t) \leq \left(X_1^{2-\beta} - (2-\beta)C_2 t \right)^{1/(2-\beta)}.$$

Hence there exists $T_X \leq X_1^{2-\beta}/(2-\beta)C_2$ such that $\gamma_X(T_X) = 0$. Note that $T_X \rightarrow 0$ as $X_1 \rightarrow 0$. On the other hand, Lemma 3.1 implies that $u(0, t) = 0$, so $\gamma_0(t) \equiv 0$ is one of solutions of (1.3). It follows from (1.4) that

$$\begin{aligned} \omega(\gamma_0(T_X), T_X) &= \omega_0(0) = 0, \\ \omega(\gamma_X(T_X), T_X) &= \omega_0(X) = |X| \neq 0. \end{aligned}$$

Since $\gamma_X(T_X) = \gamma_0(T_X) = 0$, we see that $\omega(\cdot, t)$ loses continuity at $t = T_X$.

Next we assume that $\pi < \theta < 2\pi$ and Ω is symmetric with respect to the corner. Without loss of generality, by rotation, we may assume that $\partial\Omega$ has a corner of angle θ at 0 with $\theta_0 = (\pi - \theta)/2$

in Definition 1.1. Note that Ω is symmetric with respect to the x_2 -axis. Now we consider a continuous initial data ω_0 defined by

$$(4.1) \quad \omega_0(x) = x_1.$$

Let $\widetilde{\Omega} = \{x \in \Omega : x_1 > 0\}$. Note that $\widetilde{\Omega}$ has a corner of angle $\theta/2$. Define the function $\widetilde{\omega}_0$ on $\widetilde{\Omega}$ by $\widetilde{\omega}_0 = \omega_0|_{\widetilde{\Omega}}$. In a way similar to the above argument, there is a solution $\widetilde{\omega}$ to the Euler equations (1.1) on $\widetilde{\Omega}$ such that $\widetilde{\omega}(t)$ instantaneously loses continuity in space. Now we define the function ω on Ω by $\omega(\tilde{x}) = -\omega(x)$ for $x \in \Omega$. Then ω is one of solutions to the Euler equations (1.1) in Ω with the initial data (4.1) and $\omega(t)$ instantaneously loses continuity in space. \square

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